Scattering Matrices

Bjørn Kolbrek

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When doing some work that included scattering matrices, I found that the expressions given by Kemp did not fully match up. Here follows a rederivation of the equations.

Rederivation

Going back to the fundamental equations, and assuming that eq. (6.20) is supposed to sum pressures at both sides of the discontinuity, we have

$$\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) = F\left(\mathbf{P}_+^{(2)} + \mathbf{P}_-^{(2)}\right) \tag{1}$$

$$\left(Z_c^{(2)}\right)^{-1}\left(\mathbf{P}_+^{(2)} - \mathbf{P}_-^{(2)}\right) = F^T\left(Z_c^{(1)}\right)^{-1}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{2}$$

We can rearrange to get, for pressure:

$$\left(\mathbf{P}_+^{(2)} + \mathbf{P}_-^{(2)}\right) = F^{-1}\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) \tag{3}$$

giving

$$\mathbf{P}_+^{(2)} = F^{-1}\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) - \mathbf{P}_-^{(2)} \tag{4}$$

$$\mathbf{P}_-^{(2)} = F^{-1}\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) - \mathbf{P}_+^{(2)} \tag{5}$$

and for velocity, using $\mathcal{H} = Z_c^{(2)}F^T\left(Z_c^{(1)}\right)^{-1} = \left(Y_c^{(2)}\right)^{-1}F^TY_c^{(1)}$, Y_c being the characteristic admittance $Y_c = Z_c^{-1}$:

$$\mathbf{P}_+^{(2)} - \mathbf{P}_-^{(2)} = \mathcal{H}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{6}$$

giving

$$\mathbf{P}_+^{(2)} = \mathbf{P}_-^{(2)} + \mathcal{H}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{7}$$

$$\mathbf{P}_-^{(2)} = \mathbf{P}_+^{(2)} - \mathcal{H}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{8}$$

This will produce, for $\mathbf{P}_+^{(2)}$

$$\mathbf{P}_+^{(2)} = F^{-1}\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) - \mathbf{P}_+^{(2)} + \mathcal{H}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{9}$$

$$\mathbf{P}_+^{(2)} = \frac{1}{2}\left(F^{-1} + \mathcal{H}\right)\mathbf{P}_+^{(1)} + \frac{1}{2}\left(F^{-1} - \mathcal{H}\right)\mathbf{P}_-^{(1)} \tag{10}$$

and for $\mathbf{P}_-^{(2)}$

$$\mathbf{P}_-^{(2)} = F^{-1}\left(\mathbf{P}_+^{(1)} + \mathbf{P}_-^{(1)}\right) - \mathbf{P}_-^{(2)} - \mathcal{H}\left(\mathbf{P}_+^{(1)} - \mathbf{P}_-^{(1)}\right) \tag{11}$$

$$\mathbf{P}_-^{(2)} = \frac{1}{2}\left(F^{-1} - \mathcal{H}\right)\mathbf{P}_+^{(1)} + \frac{1}{2}\left(F^{-1} + \mathcal{H}\right)\mathbf{P}_-^{(1)} \tag{12}$$

Sum and difference become

$$\begin{aligned}\mathbf{P}_+^{(2)} + \mathbf{P}_-^{(2)} = &\\ &\left\{\frac{1}{2}\left(F^{-1} + \mathcal{H}\right) + \frac{1}{2}\left(F^{-1} - \mathcal{H}\right)\right\}\mathbf{P}_+^{(1)} \\ &+ \left\{\frac{1}{2}\left(F^{-1} - \mathcal{H}\right) + \frac{1}{2}\left(F^{-1} + \mathcal{H}\right)\right\}\mathbf{P}_-^{(1)} \\ &= F^{-1}\mathbf{P}_+^{(1)} + F^{-1}\mathbf{P}_-^{(1)}\end{aligned} \tag{13}$$

$$\begin{aligned}\mathbf{P}_+^{(2)} - \mathbf{P}_-^{(2)} = &\\ &\left\{\frac{1}{2}\left(F^{-1} + \mathcal{H}\right) - \frac{1}{2}\left(F^{-1} - \mathcal{H}\right)\right\}\mathbf{P}_+^{(1)} \\ &+ \left\{\frac{1}{2}\left(F^{-1} - \mathcal{H}\right) - \frac{1}{2}\left(F^{-1} + \mathcal{H}\right)\right\}\mathbf{P}_-^{(1)} \\ &= \mathcal{H}\mathbf{P}_+^{(1)} - \mathcal{H}\mathbf{P}_-^{(1)}\end{aligned} \tag{14}$$

Expressing this in the same format as (6.24), we get

$$\begin{pmatrix} \mathbf{P}_+^{(2)} \\ \mathbf{P}_-^{(2)} \end{pmatrix} = \begin{pmatrix} \mathcal{E} & -\mathcal{F} \\ -\mathcal{F} & \mathcal{E} \end{pmatrix} \begin{pmatrix} \mathbf{P}_+^{(1)} \\ \mathbf{P}_-^{(1)} \end{pmatrix} \tag{15}$$

where

$$\mathcal{E} = \frac{1}{2}\left(\mathcal{H} + F^{-1}\right) \tag{16}$$

$$\mathcal{F} = \frac{1}{2}\left(\mathcal{H} - F^{-1}\right) \tag{17}$$

This is different from Kemp (6.24) in that the (1,2) element of equation (15) is negative, otherwise the expressions for $\mathcal{E}$ and $\mathcal{F}$ are identical.

According to Kemp eq. (5.15), for the plane wave case,

$$\left(\begin{array}{c} p_+^{(2)} \\ p_-^{(2)} \end{array} \right) = \frac{1}{1 - r_{1,2}} \left(\begin{array}{cc} 1 & -r_{1,2} \\ -r_{1,2} & 1 \end{array} \right) \left(\begin{array}{c} p_+^{(1)} \\ p_-^{(1)} \end{array} \right) \tag{18}$$

where $r_{1,2}$ is the reflection coefficient when waves are incident from side 1. Other reflection and transmission coefficients are expressed in terms of $r_{1,2}$. This supports the hypothesis that Kemp's (6.24) is wrong, and that the (1,2 and (2,1) elements of the matrix should be identical.